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Classical R-matrix theory of dispersionless systems: II. $(2+1)$ dimension theory

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Abstract
A systematic way of constructing $(2+1)$-dimensional dispersionless integrable Hamiltonian systems is presented. The method is based on the so-called central extension procedure and classical R-matrix applied to the Poisson algebras of formal Laurent series. Results are illustrated with the known and new $(2+1)$-dimensional dispersionless systems.

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1. Introduction

Dispersionless integrable Hamiltonian systems are often considered as a quasi-classical limit of the related soliton systems (see [1, 2] and the literature quoted there). Nevertheless, it seems that a more systematic approach, allowing the construction of such systems from scratch, is necessary. Actually, we are interested in a systematic way of constructing a class of dispersionless systems having a Hamiltonian structure, and infinite hierarchy of symmetries and conservation laws. One method of doing this is based on the classical R-matrix theory. As is well known, the R-matrix formalism proved very fruitful in a systematic construction of soliton systems (see for example [3–5] and the literature quoted there). So, it seems reasonable to develop such a formalism for dispersionless systems. Recently, important progress in that direction was made by Li [6]. In [7], we apply his results to a particular class of Poisson algebras [8] in order to construct multi-Hamiltonian $(1+1)$-dimensional dispersionless systems.

Having such an effective theory for constructing multi-Hamiltonian dispersionless dynamical systems in $(1+1)$ dimensions, we were prompted to extend this method to $(2+1)$ dimensions. The central extension was considered in early works by Reyman and Semenov-Tian-Shansky [9, 10] and also by Prykarpatsky [11, 12]. The central extension approach to integrable field and lattice-field systems was presented also in [13, 14].

As our construction leads, in general, to nonlocal equations, we will understand by dispersionless systems in (2 + 1) dimension PDEs of the form

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{n} v_{ij}(u, \mathcal{D})\frac{\partial u_j}{\partial x} + \sum_{j=1}^{n} w_{ij}(u, \mathcal{D})\frac{\partial u_j}{\partial y} \qquad i = 1, \ldots, n \tag{1.1}$$

where v_{ij} and w_{ij} are pseudo-differential operators of formal symbols $\mathcal{D} \equiv \partial_x^{-1}\partial_y$.

The paper is organized as follows. In section 2, we briefly present a number of basic facts and definitions of Hamiltonian dynamics on Poisson algebras concerning the formalism applied. In section 3, we present the general formulation of the central extension procedure on Poisson algebras. In sections 3 and 4, we apply this and the R-matrix procedure to the Poisson algebras of formal Laurent series. Then in section 5, we illustrate our results with the known and new integrable Hamiltonian (2 + 1)-dimensional dispersionless dynamical systems.

2. Hamiltonian dynamics on Poisson algebras: *R*-structures

Here, we repeat some basic facts presented in part I to make the paper self-consistent. The reader familiar with part I may skip this section.

Definition 2.1. *Let A be a commutative, associative algebra with unit 1. If there is a Lie bracket on A such that for each element $a \in A$, the operator* $\mathrm{ad}_a : b \mapsto [a, b]$ *is a derivation of the multiplication, then* $(A, [\cdot, \cdot])$ *is called a* Poisson algebra.

Thus, the Poisson algebras are Lie algebras with an additional associative algebra structure (with commutative multiplication and unit 1) related by the derivation property to the Lie bracket.

Let A be a Poisson algebra, A^* the dual algebra related to A by the *duality map* $\langle\cdot, \cdot\rangle \to \mathbb{R}$,

$$A^* \times A \to \mathbb{R}: \quad (\alpha, a) \mapsto \langle\alpha, a\rangle \qquad a \in A \quad \alpha \in A^* \tag{2.1}$$

and $\mathcal{D}(A^*) := \mathbb{C}^\infty(A^*)$ be a space of $\mathbb{C}^\infty$-functions on A^*. Let $F \in \mathcal{D}(A^*)$, then a map $\mathrm{d}F : A \to A$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}F(L + tL')\right|_{t=0} = \langle L', \mathrm{d}F(L)\rangle \qquad L, L' \in A^* \tag{2.2}$$

is a gradient of F.

We confine our further considerations to such Poisson algebras A for which the dual A^* can be identified with A. So, we assume the existence of a product $(\cdot, \cdot)_A$ on A which is symmetric, non-degenerate and ad-invariant:

$$(\mathrm{ad}_a b, c)_A + (b, \mathrm{ad}_a c)_A = 0 \qquad a, b, c \in A. \tag{2.3}$$

Then, we can identify A^* with A ($A^* \cong A$) by setting

$$\langle\alpha, b\rangle = (a, b)_A \qquad a, b \in A \quad \alpha \in A^* \tag{2.4}$$

where $\alpha \in A^*$ is identified with $a \in A$.

Definition 2.2. *A linear map $R : A \to A$ is called a* classical R-matrix *if the R-bracket*

$$[a, b]_R := [Ra, b] + [a, Rb] \qquad a, b \in A \tag{2.5}$$

is a second Lie product on A.

Theorem 2.3 [6]**.** *Let A be a Poisson algebra with Lie bracket $[\cdot, \cdot]$ and non-degenerate ad-invariant pairing $(\cdot, \cdot)_A$ with respect to which the operation of multiplication is symmetric,* *i.e.* $(ab, c)_A = (a, bc)_A$, $\forall a, b, c \in A$. *Assume* $R \in \mathrm{End}(A)$ *is a classical R-matrix, then for each integer* $n \geqslant -1$*, the formula*

$$\{H, F\}_n = (L, [R(L^{n+1}\,\mathrm{d}F),\ \mathrm{d}H] + [\mathrm{d}F,\ R(L^{n+1}\,\mathrm{d}H)])_A \tag{2.6}$$

where H, F *are smooth functions on* A*, defines a Poisson structure on* A*. Moreover, all* $\{\cdot,\cdot\}_n$ *are compatible.*

The related Poisson bivectors π_n are given by the following Poisson maps:

$$\pi_n : \mathrm{d}H \mapsto -\mathrm{ad}_L R(L^{n+1}\,\mathrm{d}H) - L^{n+1}R^*(\mathrm{ad}_L\,\mathrm{d}H) \qquad n \geqslant -1 \tag{2.7}$$

where the adjoint of R is defined by the relation

$$(a, Rb)_A = (R^*a, b)_A. \tag{2.8}$$

Note that the bracket (2.6) with $n = -1$ is just a Lie–Poisson bracket with respect to a Lie bracket (2.5)

$$\{H, F\}_{-1} = (L, [\mathrm{d}F,\ \mathrm{d}H]_R)_A. \tag{2.9}$$

We will look for a natural set of functions in involution with respect to the Poisson brackets (2.6). A smooth function F on A is ad-invariant if $\mathrm{d}F \in \ker \mathrm{ad}_L$, i.e

$$[\mathrm{d}F, L] = 0 \qquad L \in A \tag{2.10}$$

which are Casimir functionals of the natural Lie–Poisson bracket.

Hence, the following lemma is valid.

Lemma 2.4 [6]. *Smooth functions on* A *which are ad-invariant commute in* $\{\cdot,\cdot\}_n$*. The Hamiltonian system generated by a smooth ad-invariant function* $C(L)$ *and the Poisson structure* $\{\cdot,\cdot\}_n$ *is given by the Lax equation*

$$L_t = [R(L^{n+1}\,\mathrm{d}C), L] \qquad L \in A. \tag{2.11}$$

For any R-matrix, each two evolution equations in the hierarchy (2.11) commute due to the involutivity of the Casimir functions C_q. Each equation admits all the Casimir functions as a set of conserved quantities in involution. In this sense, we will regard (2.11) as a hierarchy of *integrable* evolution equations.

Let us assume that an appropriate product on Poisson algebra A is given by *the trace form* $\mathrm{tr} : A \to \mathbb{R}$

$$(a, b)_A = \mathrm{tr}(ab) \qquad a, b \in A. \tag{2.12}$$

To construct the simplest R-structure let us assume that the Poisson algebra A can be split into a direct sum of Lie subalgebras A_+ and A_-, i.e.

$$A = A_+ \oplus A_- \qquad [A_\pm, A_\pm] \subset A_\pm. \tag{2.13}$$

Denoting the projections onto these subalgebras by $P_\pm$, we define the R-matrix as

$$R = \tfrac{1}{2}(P_+ - P_-) \tag{2.14}$$

which is well defined.

The following two lemmas [5, 15] are useful for calculating Hamiltonians $H(L)$ from the gradients $\mathrm{d}H(L)$

Lemma 2.5 (Poincaré). *If* M *is a linear space, or more generally is of star shape* ($\forall_{L\in M}\{\lambda L : 0 \leqslant \lambda \leqslant 1\} \subset M$)*, each closed k-form is exact.*

Lemma 2.6. *Let M fulfil the condition of the Poincaré lemma. Then for an exact 1-form* $\gamma(L)$

$$H(L) = \int_0^1 \langle \gamma(\lambda L), L\rangle \,\mathrm{d}\lambda \tag{2.15}$$

is a zero-form such that $\mathrm{d}H(L) = \gamma(L)$.

Following the above scheme, we are able to construct in a systematic way integrable multi-Hamiltonian dispersionless systems, with infinite hierarchy of involutive constants of motion and infinite hierarchy of related commuting symmetries, once we fix a Poisson algebra.

3. Central extension approach

Assume now that the Poisson algebra A depends effectively on an independent parameter $y \in \mathbb{S}^1$, which naturally generates the corresponding current operator algebra $\mathcal{C}(A) = \mathcal{C}^\infty(\mathbb{S}^1, A)$ with the following modified Tr-operation:

$$\mathrm{Tr}(a) := \int_{\mathbb{S}^1} \mathrm{tr}(a)\,\mathrm{d}y \tag{3.1}$$

where tr (2.12) operation is defined for the Poisson algebra A. The scalar product reads

$$(a, b)_{\mathcal{C}(A)} := \mathrm{Tr}(a \cdot b) \tag{3.2}$$

for a and $b \in \mathcal{C}(A)$. The current Poisson algebra $\mathcal{C}(A)$ can be naturally extended via the central extension procedure: $\mathcal{C}(A) \to \overline{\mathcal{C}}(A) = \mathcal{C}(A) \otimes \mathbb{C}$ with the following Lie product:

$$[(a, \alpha), (b, \beta)] := ([a, b], \omega_2(a, b)) \tag{3.3}$$

where $\alpha, \beta \in \mathbb{C}$ and $\omega_2 : \mathcal{C}(A) \times \mathcal{C}(A) \to \mathbb{C}$ is the standard Maurer–Cartan two-cocycle on $\mathcal{C}(A)$:

$$\omega_2(a, b) := \int_{\mathbb{S}^1} \left(a, \frac{\partial b}{\partial y}\right)_A \mathrm{d}y = \mathrm{Tr}(a \cdot b_y) \qquad a, b \in \mathcal{C}(A). \tag{3.4}$$

Recall that the Maurer–Cartan two-cocycle on a Lie algebra is a bilinear $\mathbb{C}$-valued function satisfying two conditions:

(i) it is skew-symmetric

$$\omega_2(a, b) = -\omega_2(b, a) \tag{3.5}$$

(ii) it satisfies the Jacobi identity

$$\omega_2([a, b], c) + \omega_2([c, a], b) + \omega_2([b, c], a) = 0. \tag{3.6}$$

Hence, the Lie product (3.3) is well defined on $\overline{\mathcal{C}}(A)$. The scalar product on $\overline{\mathcal{C}}(A)$ is given by

$$((a, \alpha), (b, \beta))_{\overline{\mathcal{C}}(A)} := \mathrm{Tr}(a \cdot b) + \alpha \cdot \beta. \tag{3.7}$$

The Poisson bracket $\{\cdot, \cdot\}$ on the functionals $\mathcal{D}(\mathcal{C}(A))$ is defined as

$$\begin{aligned}\{H, F\}(L) &:= ((L, 1), [(\mathrm{d}F, 1), (\mathrm{d}H, 1)])_{\overline{\mathcal{C}}(A)} \\ &= ((L, 1), ([\mathrm{d}F,\ \mathrm{d}H], \omega_2(\mathrm{d}F,\ \mathrm{d}H)))_{\overline{\mathcal{C}}(A)}\end{aligned} \tag{3.8}$$

for all $(L, 1) \in \overline{\mathcal{C}}(A^*) \cong \overline{\mathcal{C}}(A)$. Then from (3.7) we get the following form:

$$\{H, F\}(L) = (L, [\mathrm{d}F,\ \mathrm{d}H])_{\mathcal{C}(A)} + \omega_2(\mathrm{d}F,\ \mathrm{d}H) \tag{3.9}$$

which can be considered as a centrally extended Lie–Poisson bracket.

Let us repeat the R-matrix approach for the current Lie algebra $\overline{\mathcal{C}}(A)$ with a natural Lie–Poisson bracket (3.9).

Lemma 3.1. *Casimir functionals $C \in \mathcal{D}(\mathcal{C}(A))$ of a Lie–Poisson bracket (3.9) satisfy the so-called Novikov–Lax equation*

$$[\mathrm{d}C, L] + (\mathrm{d}C)_y = 0 \tag{3.10}$$

for all $L \in \mathcal{C}(A^*) \cong \mathcal{C}(A)$.

Proof. For every $H, F \in \mathcal{D}(\mathcal{C}(A))$ and $L \in \mathcal{C}(A)$

$$\begin{aligned}\{H, F\}(L) &= (L, [\mathrm{d}F,\ \mathrm{d}H])_{\mathcal{C}(A)} + \omega_2(\mathrm{d}F,\ \mathrm{d}H)\\ &= (\mathrm{d}F, [\mathrm{d}H, L])_{\mathcal{C}(A)} + (\mathrm{d}F, (\mathrm{d}H)_y)_{\mathcal{C}(A)}\\ &= (\mathrm{d}F, [\mathrm{d}H, L] + (\mathrm{d}H)_y)_{\mathcal{C}(A)}\end{aligned}$$

hence for Casimir functionals $C \in \mathcal{D}(\mathcal{C}(A))$

$$\{C, F\}(L) = 0 \iff [\mathrm{d}C, L] + (\mathrm{d}C)_y = 0. \qquad \square$$

The R-structure $\overline{R} \in \operatorname{End}(\overline{\mathcal{C}}(A))$ is defined as follows:

$$[(a,\alpha), (b,\beta)]_{\overline{R}} := \left([a,b]_R, \omega_2^R(a,b)\right) \tag{3.11}$$

where $\omega_2^R(a,b) := \omega_2(Ra, b) + \omega_2(a, Rb)$. Then, the new linear Lie–Poisson bracket has the following form:

$$\begin{aligned}\{H, F\}_1(L) &= ((L,1), [(\mathrm{d}F, 1), (\mathrm{d}H, 1)]_{\overline{R}})_{\overline{\mathcal{C}}(A)}\\ &= (L, [\mathrm{d}F,\ \mathrm{d}H]_R)_{\mathcal{C}(A)} + \omega_2^R(\mathrm{d}F,\ \mathrm{d}H).\end{aligned} \tag{3.12}$$

Lemma 3.2. *The following Poisson operator is related to the linear Poisson bracket (3.12):*

$$\theta(L) : \mathrm{d}H \mapsto -\operatorname{ad}_L R\,\mathrm{d}H - R^* \operatorname{ad}_L \mathrm{d}H + (R\,\mathrm{d}H)_y + R^*(\mathrm{d}H)_y. \tag{3.13}$$

Proof. For every $H, F \in \mathcal{D}(\mathcal{C}(A))$ and $L \in \mathcal{C}(A)$

$$\begin{aligned}\{H, F\}_1(L) &= (L, [\mathrm{d}F,\ \mathrm{d}H]_R)_{\mathcal{C}(A)} + \omega_2^R(\mathrm{d}F,\ \mathrm{d}H)\\ &= (R\,\mathrm{d}F, [\mathrm{d}H, L])_{\mathcal{C}(A)} + (\mathrm{d}F, [R\,\mathrm{d}H, L])_{\mathcal{C}(A)} + (R\,\mathrm{d}F, (\mathrm{d}H)_y)_{\mathcal{C}(A)}\\ &\quad + (\mathrm{d}F, (R\,\mathrm{d}H)_y)_{\mathcal{C}(A)}\\ &= (\mathrm{d}F, -[L, R\,\mathrm{d}H] - R^*[L,\ \mathrm{d}H] + (R\,\mathrm{d}H)_y + R^*(\mathrm{d}H)_y)_{\mathcal{C}(A)}\\ &= (\mathrm{d}F, \theta(L)\,\mathrm{d}H)_{\mathcal{C}(A)}. \qquad \square\end{aligned}$$

Theorem 3.3. *The Casimir functionals $C_i \in \mathcal{D}(\mathcal{C}(A))$ of the Poisson bracket (3.8) on $\mathcal{C}(A^*) \cong \mathcal{C}(A)$ are in involution with respect to the linear Poisson bracket (3.12). Moreover, Casimir functionals C_i satisfy the following hierarchy of evolution equations:*

$$L_{t_i} = \theta(L)\,\mathrm{d}C_i = [R\,\mathrm{d}C_i, L] + (R\,\mathrm{d}C_i)_y \qquad i \in \mathbb{Z}. \tag{3.14}$$

Proof. Let C_i and $C_j \in \mathcal{D}(\mathcal{C}(A))$ be Casimir functionals, then

$$\begin{aligned}\{C_i, C_j\}_1(L) &= (L, [\mathrm{d}C_j, \mathrm{d}C_i]_R)_{\mathcal{C}(A)} + \omega_2^R(\mathrm{d}C_j, \mathrm{d}C_i)\\ &= (R\,\mathrm{d}C_j, [\mathrm{d}C_i, L] + (\mathrm{d}C_i)_y)_{\mathcal{C}(A)} + (R\,\mathrm{d}C_i, [L, \mathrm{d}C_j] - (\mathrm{d}C_j)_y)_{\mathcal{C}(A)} = 0.\end{aligned}$$

The proof of the second part of the theorem is obvious. $\square$

In a special case of Poisson algebras, which are considered in the paper, the bracket (3.12) is nothing else but a centrally extended Lie–Poisson bracket (2.9). For higher order Poisson brackets (2.6) we failed to prove the Poisson property (Jacobi identity) after central extension.

4. Poisson algebras of formal Laurent series

Let A be an algebra of Laurent series with respect to p

$$A = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x) p^i \right\} \tag{4.1}$$

where the coefficients $u_i(x)$ are smooth functions. It is obviously commutative and associative algebra under multiplication. The Lie–bracket can be introduced in infinitely many ways as

$$[L_1, L_2] = p^r \left(\frac{\partial L_1}{\partial p}\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial p} \right) := \{L_1, L_2\}_r \qquad r \in \mathbb{Z} \tag{4.2}$$

as $\mathrm{ad}_L = p^r \left(\frac{\partial L}{\partial p}\frac{\partial}{\partial x} - \frac{\partial L}{\partial x}\frac{\partial}{\partial p} \right)$ is a derivation of the multiplication, so $A_r := (A, \{\cdot,\cdot\}_r)$ are Poisson algebras. An appropriate symmetric product on A_r is given by a trace form $(a, b)_A := \mathrm{tr}(ab)$:

$$\mathrm{tr}\, L = \int_\Omega \mathrm{res}_r L \,\mathrm{d}x \qquad \mathrm{res}_r L = u_{r-1}(x) \tag{4.3}$$

which is ad-invariant. In expression (4.3) the integration denotes the equivalence class of differential expressions modulo total derivatives. For a given functional $F(L) = \int_\Omega f(u)\,\mathrm{d}x$, we define its gradient as

$$\mathrm{d}F = \frac{\delta F}{\delta L} = \sum_i \frac{\delta f}{\delta u_i} p^{r-1-i} \tag{4.4}$$

where $\delta f/\delta u_i$ is a variational derivative.

We construct the simplest R-matrix, through a decomposition of A into a direct sum of Lie subalgebras. For a fixed r let

$$\begin{aligned} A_{\geqslant -r+k} &= P_{\geqslant -r+k} A = \left\{ L = \sum_{i\geqslant -r+k} u_i(x) p^i \right\} \\ A_{<-r+k} &= P_{<-r+k} A = \left\{ L = \sum_{i<-r+k} u_i(x) p^i \right\} \end{aligned} \tag{4.5}$$

where P are appropriate projections. As we presented in [7], $A_{\geqslant -r+k}$, $A_{<-r+k}$ are Lie subalgebras in the following cases:

1. $k = 0, r = 0$,
2. $k = 1, 2, r \in \mathbb{Z}$,

which one can see through a simple inspection. Then, the R-matrix is given by the projections

$$R = \tfrac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k}) = P_{\geqslant -r+k} - \tfrac{1}{2} = \tfrac{1}{2} - P_{<-r+k}. \tag{4.6}$$

To find R^* one has to find $P^*_{\geqslant -r+k}$ and $P^*_{<-r+k}$ given by the orthogonality relations

$$(P^*_{\geqslant -r+k}, P_{<-r+k}) = (P^*_{<-r+k}, P_{\geqslant -r+k}) = 0. \tag{4.7}$$

So, we have

$$P^*_{\geqslant -r+k} = P_{<2r-k} \qquad P^*_{<-r+k} = P_{\geqslant 2r-k} \tag{4.8}$$

and then

$$R^* = \tfrac{1}{2}(P^*_{\geqslant -r+k} - P^*_{<-r+k}) = \tfrac{1}{2} - P_{\geqslant 2r-k} = P_{<2r-k} - \tfrac{1}{2}. \tag{4.9}$$

5. Centrally extended Poisson algebras of Laurent series

Let A be an algebra of Laurent series with respect to p

$$A = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x, y)p^i \right\} \tag{5.1}$$

where the coefficients $u_i(x, y)$ are smooth functions of two variables x and y. As in the $(1+1)$-dimensional case p was a conjugate coordinate related to x, let us now introduce q as a conjugate coordinate related to y. Then, introducing the extended Lie-bracket (4.2) in the form

$$\{L_1, L_2\}_r := p^r \left(\frac{\partial L_1}{\partial p}\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial p} \right) + \frac{\partial L_1}{\partial q}\frac{\partial L_2}{\partial y} - \frac{\partial L_1}{\partial y}\frac{\partial L_2}{\partial q} \qquad r \in \mathbb{Z} \tag{5.2}$$

and the extended Lax element $\mathcal{L} \equiv L - q$, $L \in A$. The Lax–Novikov equation (3.10) takes the form

$$\{\mathrm{d}C, \mathcal{L}\}_r = 0 \tag{5.3}$$

and the hierarchy of evolution equations (3.14) for Casimir functionals $C(L)$ with R-matrix given by (4.6) has the form of two equivalent representations

$$\mathcal{L}_{t_i} = \{(\mathrm{d}C_i)_{\geqslant -r+k}, \mathcal{L}\}_r = -\{(\mathrm{d}C_i)_{<-r+k}, \mathcal{L}\}_r \qquad i \in \mathbb{Z} \tag{5.4}$$

which are Lax hierarchies.

To construct dispersionless $(2+1)$-dimensional integrable equations, first we have to solve equation (5.3), which can be done by putting

$$\mathrm{d}C_i = \sum_{j\leqslant i} a_j p^j \qquad i \geqslant -r+k \tag{5.5}$$

or by

$$\mathrm{d}C_i = \sum_{j\geqslant i} a_j p^j \qquad i < -r+k \tag{5.6}$$

where the function parameters a_j are obtained from (5.3) successively via the recurrent procedure. Note that although the solutions (5.5) or (5.6) are in the form of infinite series, in fact we need only their finite parts $(\mathrm{d}C_i)_{\geqslant -r+k}$ or $(\mathrm{d}C_i)_{<-r+k}$. Hence, for a given $\mathcal{L}$, in principle, we can construct two different hierarchies of Lax equations (5.4).

We have to explain what type of Lax operator can be used in (5.4) to obtain a consistent operator evolution equivalent to some nonlinear integrable equation. Obviously, we are interested in extracting closed systems for a finite number of fields. Hence, we start by looking for Lax operators $\mathcal{L}$ in the general form

$$\mathcal{L} = u_N p^N + u_{N-1}p^{N-1} + \cdots + u_{-m+1}p^{-m+1} + u_{-m}p^{-m} - q \tag{5.7}$$

of Nth order, parametrized by finite number of fields u_i. To obtain a consistent Lax equation, the Lax operator (5.7) has to form a proper submanifold of the full Poisson algebra under consideration, i.e. the left- and right-hand sides of expression (5.4) have to lie inside this submanifold.

Observing (5.4) with some $(\mathrm{d}C)_{<-r+k} = a_{-r+k-1}p^{-r+k-1} + a_{-r+k-2}p^{-r+k-2} + \cdots$, one immediately obtains the highest order of the right-hand side of the Lax equation as

$$\begin{aligned}\mathcal{L}_t = {} & (u_N)_t p^N + (u_{N-1})_t p^{N-1} + \cdots \\ = {} & -\{(\mathrm{d}C)_{<-r+k}, u_N p^N + \text{lower}\}_r - \partial_y(\mathrm{d}C)_{<-r+k} \\ = {} & (-((-r+k-1)a_{-r+k-1}(u_N)_x - N(a_{-r+k-1})_x u_N)p^{N+k-2} + \text{lower}) \\ & + (-(a_{-r+k-1})_y p^{-r+k-1} + \text{lower})\end{aligned} \tag{5.8}$$

where lower represents lower orders. Observing (5.4) with some $(\mathrm{d}C)_{\geqslant -r+k} = \cdots + a_{-r+k+1}p^{-r+k+1} + a_{-r+k}p^{-r+k}$ one immediately obtains the lowest order of the right-hand side of the Lax equation (5.4) as

$$\begin{aligned}\mathcal{L}_t &= \cdots + (u_{-m+1})_t p^{-m+1} + (u_{-m})_t p^{-m}\\ &= \{(\mathrm{d}C)_{\geqslant -r+k}, \text{higher} + u_{-m}p^{-m}\}_r + \partial_y(\mathrm{d}C)_{\geqslant -r+k}\\ &= (\text{higher} + ((-r+k)a_{-r+k}(u_{-m})_x - (-m)(a_{-r+k})_x u_{-m})p^{-m+k-1})\\ &\quad + (\text{higher} + (a_{-r+k})_y p^{-r+k})\end{aligned} \tag{5.9}$$

where higher represents higher orders. Simple consideration of (5.8) and (5.9) with condition $N \geqslant -m$ leads to the admissible Lax polynomials with a finite number of field coordinates, which form proper submanifolds of Poisson subalgebras. They are given in the following forms:

$k = 0, r = 0$:

$$\mathcal{L} = c_N p^N + c_{N-1}p^{N-1} + u_{N-2}p^{N-2} + \cdots + u_0 - q \qquad \text{for} \quad N \geqslant 1 \tag{5.10}$$

$k = 1, r \in \mathbb{Z}$:

$$\mathcal{L} = c_N p^N + u_{N-1}p^{N-1} + \cdots + u_{-m}p^{-m} - q \qquad \text{for} \quad N \geqslant 1 - r \geqslant -m \tag{5.11}$$

$$\mathcal{L} = u_{-r}p^{-r} + u_{-r-1}p^{-r-1} + \cdots + u_{-m}p^{-m} - q \qquad \text{for} \quad -r \geqslant -m \tag{5.12}$$

$k = 2, r \in \mathbb{Z}$:

$$\mathcal{L} = u_N p^N + \cdots + u_{1-m}p^{1-m} + c_{-m}p^{-m} - q \qquad \text{for} \quad N \geqslant 1 - r \geqslant -m \tag{5.13}$$

$$\mathcal{L} = u_N p^N + \cdots + u_{3-r}p^{3-r} + u_{2-r}p^{2-r} - q \qquad \text{for} \quad N \geqslant 2 - r \tag{5.14}$$

where the u_i are dynamical fields and c_N, c_{N-1}, c_{-m} are arbitrary time-independent functions of x and y. Lax operators for $k = 0, 1, 2$: (5.10), (5.11), (5.13) form a proper submanifold in $(1+1)$ dimension [7], hence the Lax dynamics induced by them can be reduced to the $(1+1)$-dimensional space. Lax operators for $k = 1$: (5.12) and $k = 2$: (5.14) do not form a proper submanifold in $(1+1)$ dimension, hence the Lax dynamics induced by them is purely a $(2+1)$-dimensional effect, and they cannot be reduced to the $(1+1)$-dimensional space.

Hence, by knowing the restricted Lax operators $\mathcal{L}$ we can now investigate the form of gradients of Casimir functionals $\mathrm{d}C_i$ given by (5.5) or by (5.6) which satisfy equation (5.3), also some further simplest admissible reductions of Lax operators can be investigated.

The case of $k = 0$. Let us consider Lax operators of the form (5.10). Then observing (5.3) with some $\mathrm{d}C_i = a_i p^i + a_{i-1}p^{i-1} + \text{lower}$, one immediately obtains the conditions for the highest terms of $\mathrm{d}C_i$, since

$$\begin{aligned}\{a_i p^i + a_{i-1}p^{i-1} + a_{i-2}p^{i-2} + \text{lower}, \mathcal{L}\}_0 &= -N(a_i)_x c_N p^{i+N-1}\\ &\quad - (Nc_N(a_i)_x + (N-1)c_{N-1}(a_{i-1})_x)p^{i+N-2} + \text{lower} = 0.\end{aligned} \tag{5.15}$$

Therefore $(a_i)_x = (a_{i-1})_x = 0$, $ia_i(u_{N-2})_x - Nc_N(a_{i-2})_x = 0$ and so on, hence (5.5) has the following form:

$$\mathrm{d}C_i = \alpha_i p^i + \alpha_{i-1}p^{i-1} + \frac{i\alpha_i}{Nc_N}u_{N-2}p^{i-2} + a_{i-3}p^{i-3} + \text{lower} \qquad i \geqslant 0 \tag{5.16}$$

where α_i, α_{i-1} are arbitrary x-independent functions. Observing (5.3) with $\mathrm{d}C_i = \text{higher} + a_{i+1}p^{i+1} + a_i p^i$, one obtains the conditions for the lowest terms of $\mathrm{d}C_i$, since

$$\begin{aligned}&\{\text{higher} + a_{i+1}p^{i+1} + a_i p^i, \mathcal{L}\}_0 \\ &\quad = \text{higher} + ((i+1)a_{i+1}(u_0)_x + ia_i(u_1)_x - (a_i)_x u_1 + (a_i)_y)p^i + ia_i(u_0)_x p^{i-1} = 0.\end{aligned} \tag{5.17}$$

Accordingly $a_i = 0$ and $a_{i-1} = a_{i-2} = \cdots = 0$ since a_j depends linearly on $a_{j+1}, a_{j+2}, \ldots, a_i$. Hence for $k = 0$ there is only one Lax hierarchy for the $\mathrm{d}C_i$ of the form (5.16). There are not any obvious further reductions of $\mathcal{L}$.

The case of $k = 1$. For Lax operators of the form (5.11) by observing (5.3), $\mathrm{d}C_i$, given by (5.5) or (5.6), have the following forms:

$$\mathrm{d}C_i = \alpha_i p^i + \frac{i\alpha_i}{Nc_N}u_{N-1}p^{i-1} + a_{i-2}p^{i-2} + \text{lower} \qquad i \geqslant -r+1 \tag{5.18}$$

$$\mathrm{d}C_i = \text{higher} + a_{i+2}p^{i+2} + a_{i+1}p^{i+1} + \alpha_i(u_{-m})^{-\frac{i}{m}}p^i \qquad i < -r+1 \tag{5.19}$$

where α_i is an arbitrary x-independent function. For Lax operators of the form (5.12) by observing (5.3), $\mathrm{d}C_i$, given by (5.5) or (5.6), have the following forms:

$$\mathrm{d}C_i = \beta_i p^i - \partial_y^{-1}(i\beta_i(u_{-r})_x + r(\beta_i)_x u_{-r})p^{i-1} + a_{i-2}p^{i-2} + \text{lower} \qquad i \geqslant -r+1 \tag{5.20}$$

$$\mathrm{d}C_i = \text{higher} + a_{i+2}p^{i+2} + a_{i+1}p^{i+1} + \alpha_i(u_{-m})^{-\frac{i}{m}}p^i \qquad i < -r+1 \tag{5.21}$$

where α_i and β_i are arbitrary x- and y-independent functions, respectively.

We remark that, if $-m < 1-r$ in $\mathcal{L}$, there is a further admissible reduction of equations (5.4), given by $u_{-m} = 0$, since such reduced Lax polynomials are still of the form (5.11) or (5.12). We have to look for the form of gradients of Casimir functionals after such a reduction. It is easy to see that by this reduction $u_{-m} = 0$, the gradients of Casimir functionals (5.18) and (5.20) preserve the order of the highest terms, and the form. For the gradients of Casimir functionals (5.19) and (5.21) by this reduction the lowest order disappears, and as all other terms depend linearly on it, such gradients reduce to zero, except the one case $(\mathrm{d}C_i)_{<-r+1} = (L)_{<-r+1}$ which produces equation $\mathcal{L}_{t_i} = -\mathcal{L}_y$. We can continue the reductions by putting $u_{1-m} = 0$, if the reduced $\mathcal{L}$ are still of the form (5.11) or (5.12) and so on. Therefore, the reductions are proper, in general, only for the gradients of Casimir functionals in the forms (5.18) and (5.20).

The case of $k = 2$. For Lax operators of the form (5.13) by observing (5.3), $\mathrm{d}C_i$, given by (5.5) or (5.6), have the following forms:

$$\mathrm{d}C_i = \alpha_i(u_N)^{\frac{i}{N}}p^i + a_{i-1}p^{i-1} + a_{i-2}p^{i-2} + \text{lower} \qquad i \geqslant -r+2 \tag{5.22}$$

$$\mathrm{d}C_i = \text{higher} + a_{i+2}p^{i+2} - \frac{i\alpha_i}{mc_{-m}}u_{1-m}p^{i+1} + \alpha_i p^i \qquad i < -r+2 \tag{5.23}$$

where α_i is an arbitrary x-independent function. For Lax operators of the form (5.14) by observing (5.3), $\mathrm{d}C_i$, given by (5.5) or (5.6), have the following forms:

$$\mathrm{d}C_i = \alpha_i(u_N)^{\frac{i}{N}}p^i + a_{i-1}p^{i-1} + a_{i-2}p^{i-2} + \text{lower} \qquad i \geqslant -r+2 \tag{5.24}$$

$$\mathrm{d}C_i = \text{higher} - \partial_y^{-1}(i\beta_i(u_{2-r})_x - (2-r)(\beta_i)_x u_{2-r})p^{i+1} + \beta_i p^i \qquad i < -r+2 \tag{5.25}$$

where α_i and β_i are arbitrary x- and y-independent functions, respectively.

If $N > 1 - r$ in $\mathcal{L}$, there is a further admissible reduction of equations (5.4), given by $u_N = 0$ since such reduced Lax polynomials are still of the form (5.13) or (5.14). By analogous considerations as for $k = 1$, these reductions are proper, in general, only for the gradients of Casimir functionals in the forms (5.23) and (5.25).

The different schemes are interrelated as explained in the following theorem.

Theorem 5.1. *Under the transformation*

$$x' = x \qquad y' = -y \qquad p' = p^{-1} \qquad q' = q \qquad t' = t \tag{5.26}$$

the Lax hierarchy defined by $k = 1, r$ *and* $\mathcal{L}$ *transforms into the Lax hierarchy defined by* $k = 2, r' = 2 - r$ *and* $\mathcal{L}'$*, i.e.*

$$k = 1, r, \mathcal{L} \Longleftrightarrow k = 2, r' = 2 - r, \mathcal{L}'. \tag{5.27}$$

Proof. It is readily seen that the Lax operators for $k = 1$ and r of the forms (5.11) and (5.12) transform into the well-restricted Lax operators for $k = 2$ and $r' = 2 - r$ of the forms (5.13) and (5.14), respectively. Let us observe that

$$\begin{aligned}\{A, B\}_r &= p^r \left(\frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p}\right) + \frac{\partial A}{\partial q}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial q}\\ &= -p'^{-r+2}\left(\frac{\partial A'}{\partial p'}\frac{\partial B'}{\partial x'} - \frac{\partial A'}{\partial x'}\frac{\partial B'}{\partial p'}\right) - \frac{\partial A'}{\partial q'}\frac{\partial B'}{\partial y'} + \frac{\partial A'}{\partial y'}\frac{\partial B'}{\partial q'} = -\{A', B'\}'_{r'}\end{aligned}$$

and

$$(\mathrm{d}C)'_{\geqslant s} = (\mathrm{d}C')_{\leqslant -s}.$$

Hence, we have

$$\begin{aligned}\mathcal{L}_t &= \{(\mathrm{d}C)_{\geqslant -r+1}, \mathcal{L}\}_r = -\{(\mathrm{d}C)'_{\geqslant -r+1}, \mathcal{L}'\}'_{r'}\\ &= -\{(\mathrm{d}C')_{\leqslant r-1}, \mathcal{L}'\}'_{r'}\\ &= -\{(\mathrm{d}C')_{<-r'+2}, \mathcal{L}'\}'_{r'} = \mathcal{L}'_{t'}.\end{aligned}$$

□

Therefore, some dispersionless systems can be reconstructed from different Poisson algebras. Moreover, we remark that the gradients of Casimir functionals for $k = 1$ and $k = 2$ transform by $p^{-1} = p'$ reciprocally at slant, i.e. (5.18) ↔ (5.23), (5.19) ↔ (5.24) and (5.20) ↔ (5.25), (5.21) ↔ (5.24).

Two equivalent representations of Poisson structure coming from the linear Poisson tensor (3.13) with the R-matrix given by (4.6) are

$$\begin{aligned}\theta(\mathcal{L})\,\mathrm{d}H &= \{(\mathrm{d}H)_{\geqslant -r+k}, \mathcal{L}\}_r - (\{\mathrm{d}H, \mathcal{L}\}_r)_{\geqslant 2r-k}\\ &= -\{(\mathrm{d}H)_{<-r+k}, \mathcal{L}\}_r + (\{\mathrm{d}H, \mathcal{L}\}_r)_{<2r-k}.\end{aligned} \tag{5.28}$$

It turns out that the first representation yields a direct access to the lowest polynomial order of $\theta\,\mathrm{d}H$, whereas the second representation yields the information about the highest orders present. There are two options. The best situation is when a given Lax operator forms a proper submanifold of the full Poisson algebra, i.e. the image of the Poisson operator θ lies in the space tangent to this submanifold for each element. If this is not the case, *the Dirac reduction* can be invoked for the restriction of a given Poisson tensor to a suitable submanifold.

The case of $k = 0$. Let us first consider the simplest admissible Lax polynomial (5.10) of the form

$$\mathcal{L} = p^N + u_{N-2}p^{N-2} + \cdots + u_1 p + u_0 - q. \tag{5.29}$$

This is the well-known dispersionless Gelfand–Dickey case. Then, the gradient of the functional $H(\mathcal{L})$ is given in the form

$$\frac{\delta H}{\delta \mathcal{L}} = \frac{\delta H}{\delta u_0}p^{-1} + \frac{\delta H}{\delta u_1}p^{-2} + \cdots + \frac{\delta H}{\delta u_{N-2}}p^{1-N}. \tag{5.30}$$

By inserting (5.29) into (5.28) it becomes clear from the first representation of the linear tensor that the lowest order of $\theta\,\mathrm{d}H$ is at least zero, from the second representation it is evident that the highest differential order will be at most $N-2$. Hence, $\theta\,\mathrm{d}H$ is tangent to the submanifold formed by the Lax operator of the form (5.29). As a result, these Lax operators form a proper submanifold of full Poisson algebra, and the Poisson tensor, since $\left(\frac{\delta H}{\delta \mathcal{L}}\right)_{\geqslant 0} = 0$, is given by

$$\theta\left(\frac{\delta H}{\delta \mathcal{L}}\right) = \left(\left\{\mathcal{L}, \frac{\delta H}{\delta \mathcal{L}}\right\}_0\right)_{\geqslant 0}. \tag{5.31}$$

The case of $k = 1$. Let us first consider the simplest admissible Lax operator (5.11) in the form

$$\mathcal{L} = p^N + u_{N-1}p^{N-1} + \cdots + u_{1-m}p^{1-m} + u_{-m}p^{-m} - q. \tag{5.32}$$

Then the gradient of the functional $H(\mathcal{L})$ is given in the form

$$\frac{\delta H}{\delta \mathcal{L}} = \frac{\delta H}{\delta u_{-m}}p^{r+m-1} + \frac{\delta H}{\delta u_{-m+1}}p^{r+m-2} + \cdots + \frac{\delta H}{\delta u_{N-1}}p^{r-N}. \tag{5.33}$$

Inserting (5.32) into (5.28) one immediately obtains the highest and lowest order of $\theta\,\mathrm{d}H$ as

$$\begin{aligned} \theta\,\mathrm{d}H &= ((\cdots)p^{N-1} + \text{lower}) + ((\cdots)p^{2r-2} + \text{lower}) \\ &= (\text{higher} + (\cdots)p^{-m}) + (\text{higher} + (\cdots)p^{2r-1}) \end{aligned} \tag{5.34}$$

where lower (higher) represents lower (higher) orders. Hence, Lax operators of the form (5.32) form a proper submanifold for $N \geqslant 2r - 1 \geqslant -m$, as then $\theta\,\mathrm{d}H$ is tangent to this submanifold. So the linear Poisson map is

$$\theta\left(\frac{\delta H}{\delta \mathcal{L}}\right) = \left\{\left(\frac{\delta H}{\delta \mathcal{L}}\right)_{\geqslant -r+1}, \mathcal{L}\right\}_r + \left(\left\{\mathcal{L}, \frac{\delta H}{\delta \mathcal{L}}\right\}_r\right)_{\geqslant 2r-1}. \tag{5.35}$$

Otherwise a Dirac reduction is required.

Analogously, for Lax operators (5.12) in the form

$$\mathcal{L} = u_{-r}p^{-r} + u_{-r-1}p^{-r-1} + \cdots + u_{1-m}p^{1-m} + u_{-m}p^{-m} - q \tag{5.36}$$

we have

$$\begin{aligned} \theta\,\mathrm{d}H &= ((\cdots)p^{-r} + \text{lower}) + ((\cdots)p^{2r-2} + \text{lower}) \\ &= (\text{higher} + (\cdots)p^{-m}) + (\text{higher} + (\cdots)p^{2r-1}). \end{aligned} \tag{5.37}$$

Hence, this operator forms a proper submanifold for $r \leqslant 0$ and $2r - 1 \geqslant -m$. The Poisson tensor is given by (5.35). In other cases a Dirac reduction is required. The simplest case is $r = 1$ with one-field reduction. Let

$$\overline{\mathcal{L}} = u + \mathcal{L} = u + u_{-1}p^{-1} + u_{-2}p^{-2} + \cdots + u_{1-m}p^{1-m} + u_{-m}p^{-m} - q. \tag{5.38}$$

The Dirac reduction with the constraint $u = 0$ leads to the Poisson map in the form

$$\theta^{\text{red}}\left(\frac{\delta H}{\delta \mathcal{L}}\right) = \left(\left\{\frac{\delta H}{\delta \mathcal{L}}, \mathcal{L}\right\}_1\right)_{<1} + \left\{\partial_y^{-1}\operatorname{res}_1\left\{\mathcal{L}, \frac{\delta H}{\delta \mathcal{L}}\right\}_1, \mathcal{L}\right\}_1 \tag{5.39}$$

which is generally nonlocal.

The case of $k = 2$. Let us consider Lax polynomials (5.13) in the form

$$\mathcal{L} = u_N p^N + u_{N-1} p^{N-1} + \cdots + u_{1-m} p^{1-m} + p^{-m} - q. \tag{5.40}$$

Then the gradient of the functional $H(\mathcal{L})$ is given in the form

$$\frac{\delta H}{\delta \mathcal{L}} = \frac{\delta H}{\delta u_{1-m}} p^{r+m-2} + \cdots + \frac{\delta H}{\delta u_{N-1}} p^{r-N} + \frac{\delta H}{\delta u_N} p^{r-N-1}. \tag{5.41}$$

Then by analogous considerations for $k = 1$ or by theorem 5.1, $\mathcal{L}$ given by (5.40) forms a proper submanifold for $N \geqslant 2r - 3 \geqslant -m$. The Poisson tensor has the form

$$\theta\left(\frac{\delta H}{\delta \mathcal{L}}\right) = \left\{\left(\frac{\delta H}{\delta \mathcal{L}}\right)_{\geqslant -r+2}, \mathcal{L}\right\}_r + \left(\left\{\mathcal{L}, \frac{\delta H}{\delta \mathcal{L}}\right\}_r\right)_{\geqslant 2r-2}. \tag{5.42}$$

Otherwise a Dirac reduction is required.

Analogously, Lax operators (5.14) in the form

$$\mathcal{L} = u_N p^N + u_{N-1} p^{N-1} + \cdots + u_{3-r} p^{3-r} + u_{2-r} p^{2-r} - q \tag{5.43}$$

form a proper submanifold for $r \geqslant 2$ and $N \geqslant 2r - 3$. Then, the Poisson tensor has the form (5.42). Otherwise a Dirac reduction is required. The simplest case is for $r = 1$ with one-field reduction. Let

$$\overline{\mathcal{L}} = \mathcal{L} + u = u_N p^N + u_{N-1} p^{N-1} + \cdots + u_2 p^2 + u_1 p + u - q. \tag{5.44}$$

The Dirac reduction with the constraint $u = 0$ leads to the Poisson map in the form

$$\theta^{\text{red}}\left(\frac{\delta H}{\delta \mathcal{L}}\right) = \left(\left\{\mathcal{L}, \frac{\delta H}{\delta \mathcal{L}}\right\}_1\right)_{\geqslant 0} - \left\{\partial_y^{-1} \operatorname{res}_1 \left\{\mathcal{L}, \frac{\delta H}{\delta \mathcal{L}}\right\}_1, \mathcal{L}\right\}_1 \tag{5.45}$$

which is generally nonlocal.

Hence, we know the Poisson structure for $(2+1)$ dispersionless systems constructed from Poisson algebras, and since we are interested in Hamiltonian systems, we shall now consider the problem of their construction. The conserved quantities H_i are described by the Hamiltonian equations

$$\mathcal{L}_{t_i} = \theta \, \mathrm{d}H_i(\mathcal{L}). \tag{5.46}$$

First we have to find cosymmetries (1-forms) $\mathrm{d}H_i$ which are gradients of Hamiltonians. Because we are using the gradients of Casimir functionals $\mathrm{d}C_i$ to generate equations (5.4), our $\mathrm{d}H_i$ are given by projections of $\mathrm{d}C_i$ on subspaces spanned by $\mathrm{d}H_i$ in the forms (5.30), (5.33) and (5.41) for $k = 0$, 1 and 2, respectively. Then, we can apply lemma 2.6 and hence Hamiltonians are defined as follows:

$$H_i(\mathcal{L}) = \int_0^1 \operatorname{Tr}(\mathrm{d}H_i(\lambda\mathcal{L})\mathcal{L}) \, \mathrm{d}\lambda = \iint_{\Omega\times\mathbb{S}^1} \int_0^1 \operatorname{res}_r(\mathrm{d}H_i(\lambda\mathcal{L})\mathcal{L}) \, \mathrm{d}\lambda \, \mathrm{d}x \, \mathrm{d}y. \tag{5.47}$$

For Lax operator $\mathcal{L} = \sum_{i=1}^n u_i p^i - q$ the gradients from (5.47) are given by

$$\mathrm{d}H_i(\lambda\mathcal{L}) = \sum_{i=1}^n \frac{\delta h}{\delta(\lambda u_i)}(\lambda u_1, \lambda u_2, \ldots, \lambda u_n) p^{r-1-i}. \tag{5.48}$$

Hence, by using the definition of the residuum (4.4) we get that

$$\operatorname{res}_r(\mathrm{d}H_i(\lambda\mathcal{L})\mathcal{L}) = \sum_{i=1}^n u_i \frac{\delta h}{\delta(\lambda u_i)}(\lambda u_1, \lambda u_2, \ldots, \lambda u_n). \tag{5.49}$$

Contrary to the $(1+1)$-dimensional case, in the $(2+1)$ case the functional densities contain terms with x, y derivatives as well as nonlocal terms. Nevertheless, all these additional terms appear in a special form, namely they are expressed through the pseudo-differential operators of the form $\mathcal{D}^k$, $\mathcal{D}^{-k}$ where

$$\mathcal{D} := \partial_x^{-1}\partial_y \qquad \mathcal{D}^{-1} := \partial_y^{-1}\partial_x. \tag{5.50}$$

Thus, in addition to (5.47) a useful relation for the calculation of variations containing $\mathcal{D}$, derived from (2.2), is the following:

$$\frac{\delta}{\delta u}\iint_{\Omega\times\mathbb{S}^1} f(u)\mathcal{D}^k g(u)\,\mathrm{d}x\,\mathrm{d}y = \frac{\partial f(u)}{\partial u}\mathcal{D}^k g(u) + \frac{\partial g(u)}{\partial u}\mathcal{D}^k f(u). \tag{5.51}$$

6. A list of some $(2+1)$-dimensional dispersionless systems

In this section, we will display a list of the simplest nonlinear dispersionless $(2+1)$-dimensional integrable systems. Calculating the gradients $\mathrm{d}C_n$ (n-highest order) given by (5.5) we consider the Lax hierarchy

$$\mathcal{L}_{t_n} = \{(\mathrm{d}C_n)_{\geqslant -r+k}, \mathcal{L}\}_r \qquad n \in \mathbb{Z}. \tag{6.1}$$

The second hierarchy for $\mathrm{d}C_n$ given by (5.6) can be obtained by the transformation from theorem 5.1, which we leave for the interested reader. We present the Hamiltonian structure for particular choices of r. For $k = 0$ and $k = 1$ the choice $n = 1 - r$ will always lead to the dynamics $(u_i)_{t_{1-r}} = (1-r)(u_i)_x$ for the fields u_i in L, so that we may identify $t_{1-r} = \frac{1}{1-r}x$ in this case. For $(\mathrm{d}C_n)_{\geqslant -r+k} = L$ the equations become trivial, and then $\mathcal{L}_{t_{n+r-k}} = \mathcal{L}_y$. For each choice of $k = 0, 1$ or 2 and N we will exhibit the first nontrivial of the nonlinear Lax equations (6.1) associated with a chosen operator $\mathcal{L}$.

The case of $k = 0$.

Example 6.1 (The dispersionless Kadomtsev–Petviashvili: $k = 0, r = 0, N = 2.$)**.** The dispersionless Kadomtsev–Petviashvili (dKP) equation is a $(2+1)$-dimensional extension of the dispersionless KdV equation. The Lax operator for the $(2+1)$-dimensional dKP hierarchy has the form

$$\mathcal{L} = p^2 + u - q. \tag{6.2}$$

Then we derive for $(\mathrm{d}C_3)_{\geqslant 0} = p^3 + \frac{3}{2}up + \frac{3}{4}\mathcal{D}u$

$$u_{t_3} = \tfrac{3}{2}uu_x + \tfrac{3}{4}\mathcal{D}u_y = \theta\,\mathrm{d}H \tag{6.3}$$

where we get the Poisson tensor and the Hamiltonian

$$\theta = 2\partial_x \qquad H = \frac{1}{8}\iint_{\Omega\times\mathbb{S}^1}\left(u^3 + \frac{3}{2}u\mathcal{D}^2u\right)\mathrm{d}x\,\mathrm{d}y. \tag{6.4}$$

Example 6.2 (The $(2+1)$ Boussinesq hierarchy: $k = 0, r = 0, N = 3.$)**.** The Lax operator is given by

$$\mathcal{L} = p^3 + up + v - q. \tag{6.5}$$

We derive for $(\mathrm{d}C_2)_{\geqslant 0} = p^2 + \frac{2}{3}u$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ -\frac{2}{3}uu_x + \frac{2}{3}u_y \end{pmatrix} = \theta\,\mathrm{d}H. \tag{6.6}$$

Eliminating the field v from this equation we can derive the $(2+1)$-dimensional 'dispersionless' Boussinesq equation

$$u_{tt} = \tfrac{4}{3}u_{xy} - \tfrac{2}{3}(u^2)_{xx}. \tag{6.7}$$

The respective Poisson tensor and Hamiltonian are given in the following form:

$$\theta = \begin{pmatrix} 0 & 3\partial_x \\ 3\partial_x & 0 \end{pmatrix} \qquad H = \frac{1}{3}\iint_{\Omega\times\mathbb{S}^1} \left(-\frac{1}{9}u^3 + v^2 + \frac{1}{3}u\mathcal{D}u\right) \mathrm{d}x\, \mathrm{d}y. \tag{6.8}$$

Example 6.3 (The case: $k = 0, r = 0, N = 4$.)**.** The Lax operator is

$$\mathcal{L} = p^4 + up^2 + vp + w - q \tag{6.9}$$

then for $(\mathrm{d}C_2)_{\geqslant 0} = p^2 + \frac{1}{2}u$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ 2w_x - uu_x \\ \frac{1}{2}u_y - \frac{1}{2}u_x v \end{pmatrix} = \theta\, \mathrm{d}H \tag{6.10}$$

where

$$\theta = \begin{pmatrix} 0 & 0 & 4\partial_x \\ 0 & 4\partial_x & 0 \\ 4\partial_x & 0 & \partial_x u + u\partial_x \end{pmatrix} \tag{6.11}$$

$$H = \frac{1}{4}\iint_{\Omega\times\mathbb{S}^1} \left(-\frac{1}{2}u^2 v + 2vw + \frac{1}{4}u\mathcal{D}u\right) \mathrm{d}x\, \mathrm{d}y. \tag{6.12}$$

The case of $k = 1$.

Example 6.4 (Three-field hierarchy: $k = 1, r \in \mathbb{Z}\backslash 2$.)**.** The Lax operator has the form (5.11) with $N = 2 - r, m = r + 1$

$$\mathcal{L} = p^{2-r} + up^{1-r} + vp^{-r} + wp^{-r-1} - q. \tag{6.13}$$

Then for $(\mathrm{d}C_{2-r})_{\geqslant -r+1} = p^{2-r} + up^{1-r}$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} u_y + (2-r)v_x \\ ru_x v + (1-r)uv_x + (2-r)w_x \\ (1+r)u_x w + (1-r)uw_x \end{pmatrix}. \tag{6.14}$$

This Lax operator forms a proper submanifold as regards the condition $N \geqslant 2r - 1 \geqslant -m$ only for $r = 0, 1$, otherwise a Dirac reduction is required. Then for $r = 0$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} u_y + 2v_x \\ uv_x + 2w_x \\ u_x w + uw_x \end{pmatrix} = \theta\, \mathrm{d}H \tag{6.15}$$

where

$$\theta = \begin{pmatrix} 0 & 0 & 2\partial_x \\ 0 & 2\partial_x & u\partial_x - \partial_y \\ 2\partial_x & \partial_x u - \partial_y & 0 \end{pmatrix} \tag{6.16}$$

$$H = \frac{1}{16}\iint_{\Omega\times\mathbb{S}^1} \left(16vw - 2u^2\mathcal{D}v + 8u\mathcal{D}w + \frac{1}{4}u^2\mathcal{D}u^2 + 4v\mathcal{D}v \right.$$
$$\left. - u\mathcal{D}^2u^2 + 4u\mathcal{D}^2 v + u\mathcal{D}^3 u\right) \mathrm{d}x\, \mathrm{d}y. \tag{6.17}$$

For $r = 1$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = \begin{pmatrix} u_y + v_x \\ u_x v + w_x \\ 2u_x w \end{pmatrix} = \theta\, \mathrm{d}H \tag{6.18}$$

where

$$\theta = \begin{pmatrix} \partial_y & \partial_x v & 2\partial_x w \\ v\partial_x & \partial_x w + w\partial_x & 0 \\ 2w\partial_x & 0 & 0 \end{pmatrix} \qquad H = \frac{1}{2}\iint_{\Omega\times\mathbb{S}^1} (u^2 + 2v)\, \mathrm{d}x\, \mathrm{d}y. \tag{6.19}$$

Example 6.5 (Dispersionless $(2+1)$ Toda: $k = 1, r \in \mathbb{Z}\backslash\{2\}$.)**.** The first admissible reduction $w = 0$ of (6.13) leads to the two-field Lax operator

$$\mathcal{L} = p^{2-r} + up^{1-r} + vp^{-r} - q. \tag{6.20}$$

This Lax operator forms a proper submanifold only for $r = 1$, otherwise a Dirac reduction is required. Hence, for $r = 1$ by reduction $w = 0$ (6.18) we get the $(2+1)$-dimensional dispersionless Toda equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} u_y + v_x \\ u_x v \end{pmatrix} = \theta\, \mathrm{d}H \tag{6.21}$$

where

$$\theta = \begin{pmatrix} \partial_y & \partial_x v \\ v\partial_x & 0 \end{pmatrix} \qquad H = \frac{1}{2}\iint_{\Omega\times\mathbb{S}^1} (u^2 + 2v)\, \mathrm{d}x\, \mathrm{d}y \tag{6.22}$$

known up to now in a few non-Hamiltonian representations [1, 2, 17]. Changing the independent coordinate $t' = t - y$ and eliminating the u-field one gets

$$(\ln v)_{tt'} = v_{xx} \qquad \text{or} \qquad \phi_{tt'} = (\mathrm{e}^{\phi_x})_x \tag{6.23}$$

where $\phi_x = \ln v$. For $r = 0$ we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} u_y + 2v_x \\ uv_x \end{pmatrix} \tag{6.24}$$

but we lose the Hamiltonian structure since the Poisson tensor (6.16) is not reducible with the constraint $w = 0$. Hence, the Lax operator (6.20) for $r = 0$ generates equations which are non-Hamiltonian.

The next admissible reduction $w = v = 0$ of (6.14) leads to trivial equation $L_{t_{2-r}} = L_y$ since $(\mathrm{d}C_{2-r})_{\geqslant -r+1} = L$.

Example 6.6 (One-field hierarchy: $k = 1, r \in \mathbb{Z}\backslash\{2\}$.)**.** The Lax operator is given in the form

$$\mathcal{L} = p^{2-r} + (2-r)up^{1-r} - q. \tag{6.25}$$

Then one finds for $(\mathrm{d}C_{3-r})_{\geqslant -r+1} = p^{3-r} + (3-r)up^{2-r} + \left(\frac{3-r}{2-r}\mathcal{D}u + \frac{1}{2}(3-r)u^2\right) p^{1-r}$ a whole family of $(2+1)$-dimensional dispersionless one-field systems

$$u_{t_{3-r}} = -\frac{1}{2}(3-r)(1-r)u^2 u_x + \frac{r(3-r)}{2-r}uu_y + \frac{3-r}{(2-r)^2}\mathcal{D}u_y + \frac{(3-r)(1-r)}{2-r}u_x\mathcal{D}u \tag{6.26}$$

derived for the first time in [16], including the modified dKP as a special case of $r = 0$. This Lax operator forms a proper submanifold only for $r = 1$, in other cases a Dirac reduction is required. For $r = 1$ we get

$$u_{t_2} = 2uu_y + 2\mathcal{D}u_y = \theta\, \mathrm{d}H \tag{6.27}$$

where

$$\theta = \partial_y \qquad H = \iint_{\Omega\times\mathbb{S}^1} \left(\frac{1}{3}u^3 + u\mathcal{D}u\right) \mathrm{d}x\,\mathrm{d}y. \tag{6.28}$$

For $r = 0$ we get

$$u_{t_3} = -\tfrac{3}{2}u^2u_x + \tfrac{3}{4}\mathcal{D}u_y + \tfrac{3}{2}u_x\mathcal{D}u \tag{6.29}$$

and by Dirac reduction of (6.16) with the constraint $w = v = 0$ we get the formal Poisson tensor

$$\theta^{\mathrm{red}} = 8\partial_x(\partial_y - 2u\partial_x)^{-1}\partial_x(\partial_y - 2\partial_x u)^{-1}\partial_x \tag{6.30}$$

and the related symplectic tensor

$$J = (\theta^{\mathrm{red}})^{-1} = \tfrac{1}{8}(\mathcal{D} - 2u)\partial_x^{-1}(\mathcal{D} - 2u) \tag{6.31}$$

such that $Ju_{t_3} = \mathrm{d}H$, where

$$H = \frac{3}{32}\iint_{\Omega\times\mathbb{S}^1}\left(-\frac{1}{3}u^6 + u\mathcal{D}u^4 + \frac{1}{2}u^2\mathcal{D}^2u^2 - u^2(\mathcal{D}u)^2 \right.$$
$$\left. + \frac{1}{3}(\mathcal{D}u)^3 - u\mathcal{D}^3u^2 + \frac{1}{2}u\mathcal{D}^4u\right)\mathrm{d}x\,\mathrm{d}y. \tag{6.32}$$

Example 6.7 (Three-field hierarchy: $k = 1, r \in \mathbb{Z}$.)**.** This case does not exist in $(1+1)$ dimension. The Lax operator has the form (5.12) with $m = r + 2$

$$\mathcal{L} = up^{-r} + vp^{-r-1} + wp^{-r-2} - q. \tag{6.33}$$

Then for $(\mathrm{d}C_{2-r})_{\geqslant -r+1} = p^{2-r} + (r-2)\mathcal{D}^{-1}up^{1-r}$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = (r-2)\begin{pmatrix} ru\mathcal{D}^{-1}u_x + (1-r)u_x\mathcal{D}^{-1}u - v_x \\ (1+r)v\mathcal{D}^{-1}u_x + (1-r)v_x\mathcal{D}^{-1}u - w_x \\ (2+r)w\mathcal{D}^{-1}u_x + (1-r)w_x\mathcal{D}^{-1}u \end{pmatrix}. \tag{6.34}$$

This Lax operator forms a proper submanifold as regards the condition $2r - 1 \geqslant -m$ only for $r = 0$, otherwise a Dirac reduction is required. Then for $r = 0$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = -2\begin{pmatrix} u_x\mathcal{D}^{-1}u - v_x \\ v\mathcal{D}^{-1}u_x + v_x\mathcal{D}^{-1}u - w_x \\ 2w\mathcal{D}^{-1}u_x + w_x\mathcal{D}^{-1}u \end{pmatrix} = \theta\,\mathrm{d}H \tag{6.35}$$

where

$$\theta = \begin{pmatrix} 0 & -\partial_y & 0 \\ -\partial_y & 0 & 0 \\ 0 & 0 & \partial_x w + w\partial_x \end{pmatrix} \tag{6.36}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} (-2u\mathcal{D}^{-1}w - v\mathcal{D}^{-1}v + v(\mathcal{D}^{-1}u)^2)\,\mathrm{d}x\,\mathrm{d}y. \tag{6.37}$$

For $r = 1$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = -\begin{pmatrix} u\mathcal{D}^{-1}u_x - v_x \\ 2v\mathcal{D}^{-1}u_x - w_x \\ 3w\mathcal{D}^{-1}u_x \end{pmatrix} = \theta\,\mathrm{d}H. \tag{6.38}$$

We derive the Poisson tensor from (5.39), then

$$\theta^{\mathrm{red}} = \begin{pmatrix} -u\mathcal{D}^{-1}\partial_x u + \partial_x v + v\partial_x & -2u\mathcal{D}^{-1}\partial_x v + 2\partial_x w + w\partial_x & -3u\mathcal{D}^{-1}\partial_x w \\ -2v\mathcal{D}^{-1}\partial_x u + \partial_x w + 2w\partial_x & -4v\mathcal{D}^{-1}\partial_x v & -6v\mathcal{D}^{-1}\partial_x w \\ -3w\mathcal{D}^{-1}\partial_x u & -6w\mathcal{D}^{-1}\partial_x v & -9w\mathcal{D}^{-1}\partial_x w \end{pmatrix} \tag{6.39}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} u\,\mathrm{d}x\,\mathrm{d}y. \tag{6.40}$$

Example 6.8 (Two-field hierarchy: $k = 1, r \in \mathbb{Z}$.)**.** The first admissible reduction $w = 0$ of (6.33) leads to the two-field Lax operator

$$\mathcal{L} = up^{-r} + vp^{-r-1} - q. \tag{6.41}$$

This Lax operator forms a proper submanifold only for $r = 0$, otherwise a Dirac reduction is required. Hence, for $r = 0$ by reduction $w = 0$ of (6.35) we get

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = -2\begin{pmatrix} u_x\mathcal{D}^{-1}u - v_x \\ v\mathcal{D}^{-1}u_x + v_x\mathcal{D}^{-1}u \end{pmatrix} = \theta\,\mathrm{d}H \tag{6.42}$$

where

$$\theta = \begin{pmatrix} 0 & -\partial_y \\ -\partial_y & 0 \end{pmatrix} \tag{6.43}$$

$$H = \frac{2}{3}\iint_{\Omega\times\mathbb{S}^1} (-v\mathcal{D}^{-1}v + v(\mathcal{D}^{-1}u)^2)\,\mathrm{d}x\,\mathrm{d}y. \tag{6.44}$$

For $r = 1$ we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = -\begin{pmatrix} u\mathcal{D}^{-1}u_x - v_x \\ 2v\mathcal{D}^{-1}u_x \end{pmatrix} = \theta\,\mathrm{d}H. \tag{6.45}$$

We derive the Poisson tensor from (6.39) with the constraint $w = 0$, then

$$\theta^{\mathrm{red}} = \begin{pmatrix} -u\mathcal{D}^{-1}\partial_x u + \partial_x v + v\partial_x & -2u\mathcal{D}^{-1}\partial_x v \\ -2v\mathcal{D}^{-1}\partial_x u & -4v\mathcal{D}^{-1}\partial_x v \end{pmatrix} \tag{6.46}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} u\,\mathrm{d}x\,\mathrm{d}y. \tag{6.47}$$

Example 6.9 (One-field hierarchy: $k = 1, r \in \mathbb{Z}$.)**.** The second admissible reduction $w = v = 0$ of (6.33) leads to the one-field Lax operator

$$\mathcal{L} = up^{-r} - q. \tag{6.48}$$

This Lax operator does not form a proper submanifold as the condition $2r - 1 \geqslant -m$ is violated, hence a Dirac reduction is required. For $r = 0$ by reduction $v = w = 0$ of (6.35) we get

$$u_{t_2} = -2u_x\mathcal{D}^{-1}u \tag{6.49}$$

but we lose the Hamiltonian structure as the Poisson tensor (6.43) is not Dirac reducible with constraint $v = w = 0$. For $r = 1$ we have

$$u_{t_1} = -u\mathcal{D}^{-1}u_x = \theta\,\mathrm{d}H. \tag{6.50}$$

We derive the Poisson tensor from (6.46) with the constraint $w = 0$, then

$$\theta^{\mathrm{red}} = -u\mathcal{D}^{-1}\partial_x u \tag{6.51}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} u\,\mathrm{d}x\,\mathrm{d}y. \tag{6.52}$$

The case of $k = 2$.

Example 6.10 (One-field hierarchy: $k = 2, r \in \mathbb{Z}\backslash\{2\}$.)**.** The simplest admissible Lax operator is given by

$$\mathcal{L} = u^{2-r} p^{2-r} - q. \tag{6.53}$$

This case does not exist in $(1+1)$ dimension. In this case we have to consider separately two cases: $r \neq 1$ and $r = 1$. Then, one finds again a whole family of $(2+1)$-dimensional dispersionless one-field systems [16] including a dispersionless $(2+1)$-dimensional Harry Dym equation as a special case of $r = 0$:

$r \neq 1$:

For

$$(\mathrm{d}C_{3-r})_{\geqslant -r+2} = u^{3-r} p^{3-r} + \frac{3-r}{(r-1)(2-r)} u^{2-r} \mathcal{D} u^{r-1} p^{2-r}$$

one finds

$$u_{t_{3-r}} = \frac{3-r}{(r-1)(2-r)} u_y \mathcal{D} u^{r-1} + \frac{3-r}{(2-r)^2} u \mathcal{D} u^{r-2} u_y. \tag{6.54}$$

$r = 1$:

For

$$(\mathrm{d}C_2)_{\geqslant 1} = u^2 p^2 + 2u\mathcal{D}(\ln u) p$$

one finds

$$u_{t_2} = 2u_y \mathcal{D} \ln u + 2u\mathcal{D}(\ln u)_y. \tag{6.55}$$

To get θ, we have to make a Dirac reduction as the conditions $r \geqslant 2$, $N \geqslant 2r - 3$ are violated. The Poisson tensor for $r = 1$ is given by (5.45), then we get for (6.55) the Hamiltonian structure, where

$$\theta^{\mathrm{red}} = u\mathcal{D}^{-1}\partial_x u \qquad H = \iint_{\Omega\times\mathbb{S}^1} \left(\ln u \mathcal{D}^3 \ln u + \frac{1}{3} (\mathcal{D} \ln u)^3 \right) \mathrm{d}x\, \mathrm{d}y. \tag{6.56}$$

Example 6.11 (Two-field hierarchy: $k = 2, r \in \mathbb{Z}\backslash\{3\}$.)**.** The Lax operator is given by

$$L = up^{3-r} + vp^{2-r} - q. \tag{6.57}$$

This case is nonreducible to $(1+1)$ dimension. Then, one finds for $(\mathrm{d}C_{2-r})_{\geqslant -r+2} = u^{\frac{2-r}{3-r}} p^{2-r}$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_{2-r}} = \frac{2-r}{3-r} u^{\frac{-1}{3-r}} \begin{pmatrix} (3-r)uv_x - (2-r)u_x v \\ u_y \end{pmatrix}. \tag{6.58}$$

To get θ we have to make a Dirac reduction as the conditions $r \geqslant 2$, $N \geqslant 2r - 3$ are violated. The Poisson tensor for $r = 1$ is given by (5.45), then

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \frac{\sqrt{u}}{2u} \begin{pmatrix} 2uv_x - u_x v \\ u_y \end{pmatrix} = \theta^{\mathrm{red}}\, \mathrm{d}H \tag{6.59}$$

where

$$\theta^{\mathrm{red}}\, \mathrm{d}H = \begin{pmatrix} 4u\mathcal{D}^{-1}\partial_x u & 2u\mathcal{D}^{-1}\partial_x v \\ 2v\mathcal{D}^{-1}\partial_x u & 2v\mathcal{D}^{-1}\partial_x v + \partial_x u + u\partial_x \end{pmatrix} \tag{6.60}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} \left(-\frac{1}{8}\frac{v^3\sqrt{u}}{u^2} - \frac{1}{3}u\mathcal{D}\frac{v\sqrt{u}}{u} + \frac{3}{4}\frac{v\sqrt{u}}{u}\mathcal{D}\ln u \right) \mathrm{d}x\,\mathrm{d}y. \tag{6.61}$$

Example 6.12 (Two-field hierarchy: $k = 2, r \in \mathbb{Z}\backslash\{2\}$.)**.** The Lax operator is given by

$$\mathcal{L} = u^{2-r}p^{2-r} + vp^{1-r} + p^{-r} - q. \tag{6.62}$$

Then, for $(\mathrm{d}C_{2-r})_{\geqslant -r+2} = u^{2-r}p^{2-r}$ one finds

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} u_y - (1-r)u_x v + u v_x \\ (2-r)ru^{1-r}u_x \end{pmatrix}. \tag{6.63}$$

This Lax operator forms a proper submanifold only for $r = 1$, otherwise a Dirac reduction is required. Hence, for $r = 1$ we get

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} u_y + u v_x \\ u_x \end{pmatrix} = \theta\,\mathrm{d}H \tag{6.64}$$

where

$$\theta\,\mathrm{d}H = \begin{pmatrix} 0 & u\partial_x \\ \partial_x u & -\partial_y \end{pmatrix} \tag{6.65}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} \left(u + \frac{1}{2}v^2 + v\mathcal{D}\ln u + \frac{1}{2}\ln u\,\mathcal{D}^2\ln u \right) \mathrm{d}x\,\mathrm{d}y. \tag{6.66}$$

Example 6.13 (Three-field hierarchy: $k = 2, r \in \mathbb{Z}$.)**.** The Lax operator is given by

$$\mathcal{L} = up^{2-r} + vp^{1-r} + wp^{-r} + p^{-r-1} - q. \tag{6.67}$$

Then for $(\mathrm{d}C_{2-r})_{\geqslant -r+2} = up^{2-r}$ one finds

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} u_y - (1-r)u_x v + (2-r)u v_x \\ r u_x w + (2-r)u w_x \\ (1+r)u_x \end{pmatrix}. \tag{6.68}$$

This Lax operator forms a proper submanifold only for $r = 1$, otherwise a Dirac reduction is required. Hence, for $r = 1$ we get

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = \begin{pmatrix} u_y + u v_x \\ u_x w + u w_x \\ 2u_x \end{pmatrix} = \theta\,\mathrm{d}H \tag{6.69}$$

where

$$\theta = \begin{pmatrix} 0 & u\partial_x & 0 \\ \partial_x u & -\partial_y & 0 \\ 0 & 0 & 2\partial_x \end{pmatrix} \tag{6.70}$$

and

$$H = \iint_{\Omega\times\mathbb{S}^1} \left(\frac{1}{2}v^2 + uw + v\mathcal{D}\ln u + \frac{1}{2}\ln u\,\mathcal{D}^2\ln u \right) \mathrm{d}x\,\mathrm{d}y. \tag{6.71}$$

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